simple expedient of purifying the water supply.

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SCIENTIFIC BOOKS

Lectures on the Differential Geometry of Curves and Surfaces. By A. R. FORSYTH. University Press, Cambridge, 1912. Large octavo. Pp. xxiii + 585. Price, $8.

Professor Forsyth's skill and versatility in the writing of mathematical treatises, already proved by his well-known works on differential equations and the theory of functions, is again illustrated by this new volume, his first in the field of geometry. The lectures were delivered, in substantially their present form, during the author's tenure of the Sadlerian professorship at Cambridge. They make very interesting reading. The style is graceful, and the technical discussions are illuminated by many passages on the history and development of the special topics considered.

Naturally no attempt is made to cover the whole field of differential geometry. Not even the classic four-volume treatise of Darboux pretends to include all the applications of the methods of the infinitesimal calculus to the domain of geometry. In particular, the author omits all extensions to hyperspace and non-Euclidean geometrics. His main aim is to "expound those elements with which eager and enterprising students should become acquainted," and to provide such students, who, later, may devote themselves to original work "with some of the instruments of research."

The author restricts himself to curves and surfaces in ordinary Euclidean space and uses the direct methods introduced by Gauss. "I have made no attempt to give what could only have been a rather faint reproduction of Darboux's treatment, which centers round the tri-rectangular trihedron at any point of a curve or surface or system. My hope is that students may experience an added stimulus when they find that different methods combine in the development of growing knowledge." It must be admitted that, by showing the power of the more natural methods (combined, of course, with typical Cambridge skill in analytical manipulation) in the solution of extremely difficult problems, the author's procedure is amply justified.

As regards logical rigor the work is about on a level with the texts of Bianchi, Scheffers, and Eisenhart. No attempt is made to lay precise function-theoretic foundations for the geometric structure which is erected. In particular the concepts of analytic curve and surface, employed throughout the work, are never formulated precisely. Professor Study's vigorous criticism of the new edition of Bianchi in this aspect applies in fact to all standard treatises on differential geometry. It must be confessed that the claims of rigor are not emphasized in geometry to nearly the same extent as in analysis. In this respect geometry in fact occupies a position between analysis and physics, and to that extent belongs to applied rather than to pure mathematics. Study has himself outlined a proper basis for the treatment of analytic curves,[1] but this has not yet been digested into a form suitable for an introductory text, and the corresponding discussion of surfaces is still to be undertaken. No doubt, in the future—how near one can not say—Study's high and beautiful ideal will become realized. Meanwhile, most geometers, at least when they write on differential geometry, follow the older and what they considered the most expedient approach. Perhaps a distinction should be made, even in the domain of graduate mathematics, between *pedagogic* books and *logical* books. The evolution toward a rigorous treatment (never perfect, but at least up to the highest standard of a given period of mathematics) is obviously inevitable.

As regards the introduction of *imaginary* configurations in geometry the author follows the traditional half-hearted policy of considering them only when it is convenient, or at least traditional, to do so. Thus, in connection with a real surface, it is analytically expedient to introduce certain curves, of course imaginary, whose length (between any two points) is zero. [These the author desig-

[1] In two memoirs published in the *Trans. Amer. Math. Soc.*, 1909, 1910.

nates as *nul* lines, a very good name in itself, but already in use in an entirely different sense in connection with the so-called nul-system of statics and line geometry, and furthermore unnecessary since the term *minimal* lines is quite standardized in the literature.] Again in connection with the problem of geodesic representation, the usual discussion of the real solutions of Beltrami and Dini is followed by the imaginary solutions discovered by Sophus Lie. If then imaginary figures are allowed even when, as in this last instance, they are, however interesting, really inconvenient, why not, as Study advocates, introduce them deliberately and systematically?

The student usually gets the impression that whatever is true in the real domain, will, by some very nebulous principle of continuity, also be true in the larger complex (real and imaginary) domain. This is actually the case in a remarkably large number of questions, but certainly not in all. The exceptional character of minimal curves has long been recognized, but only recently the peculiar curves lying in a minimal plane have been investigated by Study and his students.

Even in the domain of curves lying in an ordinary (Euclidean) plane, the reviewer has recently encountered a striking instance of how imaginary figures may have essentially different properties from real figures. It is a standard theorem in elementary calculus that when one point approaches another on a curve the arc and the chord becomes ultimately equal, that is, the ratio of the arc to the chord approaches unity as its limit. This property of real analytic curves is true of most imaginary curves, as can be verified by calculation, but not of all. In the simplest class of exceptional imaginary curves, the limit is a certain irrational number, approximately .94. Thus the arc, instead of becoming equal to the chord, becomes *less* (of course in absolute value, since both arc and chord are complex quantities). Again, the statement is usually made that the difference between the arc and the chord is an infinitesimal of *third* order; but in the present instance it is in fact of the *first* order. The same class of curves shows that even when there is no cusp or singular point the radius of curvature may vanish. It is remarkable that whenever the limit mentioned is not unity, it is at most equal to .94. For space curves the result is quite different, since the limit may then be any number, real or imaginary.[2]

The moral of all this is that if, originally, imaginaries were introduced into geometry because they made the statement of propositions, especially of algebraic geometry, *easier*, and bore out the principle of continuity, we have to pay for this nowadays by a systematic treatment of the imaginary figures in complete generality. We must look, with an enlightened view, over the entire complex domain, instead of restricting our attention, from some more or less accidental motive, to some cross section connected, more or less closely, with the original real domain. Of course there will always be justification for a purely real geometry (as instanced say by analysis situs, or the geometry of connection, including the theory of knots); but differential geometry has been guided mainly by the theory of analytic functions (power series), rather than the theory of functions of a real variable, and the tendency toward a perfect correspondence with the former theory seems all-compelling. The present period is one of transition, and that is always hard on both the writer of text-books and his students.

The author has certainly succeeded in getting into one volume most of the more important standard topics. This is shown by the chapter headings: Curves in space, General theory of surfaces, Organic curves of a surface, Lines of curvature, Geodesics, General curves on a surface and differential invariants, Comparison of surfaces, Minimal surfaces, Surfaces with plane or spherical lines of curvature and Weingarten surfaces, Deformation of surfaces, Triply-orthogonal systems of surfaces, Congruences of curves.

The treatment of each of these topics is quite elaborate, in many instances the proofs are more elegant than those usually given, and an abundant selection of problems (many

[2] See *Bull. Amer. Math. Soc.*, Vol. 20, p. 727.

of them difficult, and often containing important results, as is to be expected of a Cambridge treatise) is included.

Perhaps the most important and interesting feature of the book is the long chapter dealing with differential invariants, covariants, and parameters. The algebraic method employed is due to Forsyth himself, and was first published in a memoir in the *Philosophical Transactions*, 1903. The calculations are arranged very ingeniously, and the detailed results are certainly useful. The geometric interpretations, however, are not always clear, and sometimes they appear to be only partly geometric. The practise of calling a result geometric when it is in fact semi-geometric and semi-algebraic is unfortunately rather prevalent.

The author's terminology, in this connection, can not be recommended. He speaks of the invariants of "a single curve," when he really means a *system* of curves, simply infinite in number. His results have in fact no meaning for a single curve. It would be absurd to imply that the author's ideas are not clear—it is merely a matter of careless terminology. The distinction between a *system of curves* and a *curve* is just as great as that between a *curve* and a *point*. Of course a system is made up of an infinitude of curves, just as a curve is made up of an infinitude of points, but that is no excuse for identifying the configurations.

The long list of invariants for "two curves" refers actually to two simply infinite systems of curves, a figure usually called a *net* of curves. The author is not here discussing doubly-infinite systems. It is best to avoid the ambiguous term double system: it refers sometimes to a double infinity (that is, infinity times infinity) and sometimes to two single infinities (that is, a net).

After the discussion of "one curve" and "two curves," the rather mysterious statement is made that "we could not consider profitably more than two independent curves." As a matter of fact there are some very important (and naturally very difficult) problems connected especially with *three* systems, and, in the reviewer's opinion, the investigation will have to be extended.

Another more serious confusion of terms, and even of ideas, is prevalent in geometric literature. We refer to the distinction between a *system of curves* and a *parametered system of curves*. The latter object arises frequently in applications. For example, in a topographic map we have to deal not merely with the system of contour lines, but with the particular numbers attached to these curves indicating the heights above sea level. The same system of curves with different numbers would represent a different topography. In most discussions in geometry we are concerned merely with the system of curves; but if the attached numbers or parameters are also of significance, as they often are, the compound object should be called not a system, but a parametered system.

Even in one dimension an analogous distinction is important. A curve consists of a single infinity of points: if the points are labeled with numbers, then we have a new figure, a *parametered curve*. A straight line with a logarithmic scale is certainly different from a straight line with an ordinary uniform scale. A correct and well developed terminology is at hand in, for example, d'Ocagne's *Nomography*. This branch of mathematics was originated and developed almost entirely by engineers (mainly the French school), rather than pure mathematicians, but it is certainly time for writers on geometry to take advantage of their work.

With this terminology, it is possible to state very compactly a fundamental theorem in the theory of functions of a complex variable, as follows: Any (analytic) parametered curve can be converted into any second parametered curve by a unique direct (and a unique reverse) conformal transformation. This is true of parametered curves but not of curves: any curve can be converted into a second curve by an infinitude of conformal transformations.

The author is to be commended for not confining himself, as much as most writers do, to questions of first and second order (of

geometric infinitesimals). Third order questions have been treated in a haphazard way in the standard literature. Even in the simple case of plane curves, the average student becomes familiar only with the interpretation of the first derivative as slope (tangent line), and of the second derivative as curvature (osculating curve). As regards the third derivative, his mind is usually blank. Even the elementary books should contain the definition of *deviation,* introduced by Transon over seventy years ago.

An excellent index and table of symbols will be appreciated by the student, and make the volume serviceable for convenient reference. The press work throughout is quite perfect.

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A Catalogue of the Fishes of Japan. By DAVID STARR JORDAN, SHIGEHO TANAKA and JOHN OTTERBEIN SNYDER. Journal of the College of Science, Tokyo Imperial University, Vol. XXXIII., article 1. Tokyo, 1913. 8vo. Pp. 1–497, with 396 text-figures.

Japan possesses a wonderfully rich fish fauna. This is due to several causes: first, to the fact that she consists of a chain of islands with innumerable small, sheltered bodies of water which afford great variety of depth, physical condition of sea floor, etc.,—factors highly favorably to a diversity of fish life. Secondly, to her remarkable north-and-south extent, which gives her in addition to the regular north temperate fauna, in itself unusually rich in this instance, a subtropical fauna allied to that of the Philippine Islands, in the south, and a fauna merging into a subarctic one, in the extreme north. Thirdly, her eastern coast is touched by the *Kuroshiwo,* or warm black current, which harbors many tropical forms, some of them exceedingly rare, or in fact, only known from this current.

With such remarkable conditions, it is not surprising that ichthyologists should have been attracted to the study of Japanese fishes. Many of the writers of this and the preceding generation have taken a hand in describing portions of this fauna as materials were brought from Japan, so that an extensive literature has grown up about it. And there has been at least one extensive work on this fauna—that of Temminck and Schlegel, in two superb folio volumes, one of text and one of plates, published between 1842 and 1850.

But an entirely new chapter in Japanese ichthyology was opened when, in 1900, Chancellor Jordan and Professor Snyder, of Leland Stanford University, visited Japan for the purpose of studying the fishes. As a result of the collections then made, and of others made subsequently, including one by Gilbert and Snyder in the *Albatross,* in 1906, Jordan and his associates Gilbert, Snyder, Starks, Richardson, Fowler, Herre, Seale and Thompson, have worked unremittingly on this fauna, publishing paper after paper, until a long series, numbering several score, has now appeared. They have described hundreds of new species; figured, revised, re-studied, and thrown light on many of the darker problems relating to the fishes of Japan.

Early in the course of these studies it became patent to Jordan and Snyder that it was necessary to take stock of what had already been done on the Japanese fauna. Accordingly, in 1901, they published "A preliminary check-list of the fishes of Japan." This incorporated all the data then available, including two lists published by Japanese ichthyologists. The number of species listed was 686, many, however, only doubtfully referred to Japan. And now we have a new catalogue of the fishes of Japan from the pen of Jordan, Tanaka and Snyder. An idea of the enormous wealth of the Japanese fish fauna, as well as of the great stride that has been made in its study in a little over a decade, is shown in the fact that the present catalogue lists no less than 1,236 species (including the 6 given in the Additions and Corrections, pp. 429–430), or nearly twice the number known in 1901.

The catalogue—or check-list, as it might more correctly have been termed—enumerates the families, genera and species of the fishes occurring in the waters of Japan. Under each species is given a reference to the first describer, and generally, to a reviser; together